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A generalised form of extended lattice–lattice scaling and its relationship to the scaled equation of state with applications to the Ising model

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Abstract. A natural generalisation of lattice–lattice scaling is suggested which holds exactly for the second most singular amplitudes for the Ising model on the triangular, honeycomb, square *and* Kagomé lattices. In general, this scaling theory requires three scaling parameters g, n and m, but includes extended lattice–lattice scaling as the special case of $m = n$. The generalised form of lattice–lattice scaling does not seem to be applicable to the three-dimensional Ising model. The connections between lattice–lattice scaling, its extension and generalisation, and the critical equation of state including correction terms are established and discussed.

1. Introduction and summary

It is known that while extended lattice–lattice (ELL) scaling holds for the second most singular amplitudes for the simple Ising model on the triangular (T), honeycomb (H) and square (S) lattices (Guttmann 1974), it does not hold for the Kagomé (K) lattice (Ritchie and Betts 1975). It is the aim of this paper to introduce a generalised extended lattice–lattice (GELL) scaling theory which holds for all four lattices and which includes ELL scaling as a special case, and also to discuss the connections between these types of scaling and the critical equation of state including correction terms.

ELL scaling theory is also known to break down for the three-dimensional Ising model (Oitmaa and Ho-Ting-Hun 1976, Guttmann 1977). We have applied our GELL scaling to the three-dimensional lattices and find that this also fails to hold, at least to within the numerical uncertainties of the amplitudes. Either our uncertainties are unrealistically small or, more likely, the second most singular amplitudes for the three-dimensional Ising model do not scale, at least not in any obvious or simple way. This breakdown of ELL scaling and GELL scaling appears to be the case for the Ising model in the Bethe approximation and also for the spherical model (see § 6). Consequently, we concentrate on the two-dimensional Ising lattices in the remainder of this paper.

Our generalisation of ELL scaling is suggested by first determining in § 2 the implications of lattice–lattice (LL) scaling (Betts *et al* 1971) for the scaling function $h_0(x)$ in the critical equation of state (Griffiths 1967, Gaunt and Domb 1970),

$$h \equiv mH/kT = M^{\delta}h_0(x). \tag{1.1}$$

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Here

$$x=[(T-T_c)/T]\,M^{-1/\beta}\equiv tM^{-1/\beta} \tag{1.2}$$

and all the other symbols take their usual meaning (Gaunt and Domb 1970). It turns out that LL scaling of the most singular amplitudes implies $h_0(x)/h_0(0)$ is the same function of x/x_0 for all lattices of given dimensionality; that is, $h_0(x)/h_0(0)$ is a *universal* function of x/x_0. We remind the reader (see Betts *et al* 1971) that LL scaling is a weaker form of scaling than that embodied in (1.1); the latter also implies the usual critical exponent relations including, in particular, exponent symmetry. However, for the two-dimensional Ising model which is our main concern here, all these exponent relations are known to be satisfied exactly.

Correction terms to (1.1) have been calculated by Domb (Domb 1971, Domb and Gaunt 1971) and take the form

$$h=M^{\delta}h_0(x)+M^{\delta+(1/\beta)}h_1(x)+M^{\delta+(2/\beta)}h_2(x)+\ldots+M^{\delta+(\gamma/\beta)}k_1(x)$$
$$+M^{\delta+[(\gamma+1)/\beta]}k_2(x)+\ldots+\ldots. \tag{1.3}$$

It is easily shown that for the two-dimensional Ising model this equation is consistent with that implied by Wegner's work using renormalisation group arguments (Wegner 1972). The derivation of (1.3) assumes that the high-temperature behaviour of the field derivatives of the free energy evaluated in zero field are of Darboux form, that is, the correction to scaling exponent is unity. This seems to be a not unreasonable assumption at least for the two-dimensional Ising model where at least the zeroth and second field derivatives are known rigorously to be of this form (Onsager 1944, Barouch *et al* 1973). For the three-dimensional Ising model, the leading correction term for the high-temperature initial susceptibility appears to take the non-Darboux form (Wortis 1970 *Newport Conference on Phase Transitions* unpublished, Wegner 1972)

$$(kT/m^2)\chi_0\sim C^+_{0,2}t^{-\gamma}(1+Et^{\Delta_1}+Ft+\ldots), \qquad (H=0,\ t\to 0+) \tag{1.4}$$

where the most recent estimate of the leading correction to scaling exponent is $\Delta_1=0{\cdot}493\pm0{\cdot}007$ (Le Guillou and Zinn-Justin 1977). However, the critical amplitudes $C^+_{0,2}$, E and F are all spin dependent and the best available evidence (Camp and Van Dyke 1975) suggests that $E=0$ when $S=\frac{1}{2}$. In this event, (1.4) reduces to Darboux form at least to leading order. The same probably happens for the higher derivatives suggesting that (1.3) may also be appropriate for the $S=\frac{1}{2}$ three-dimensional Ising model.

In § 3 we make the not unreasonable assumption that the leading correction function $h_1(x)$ scales in a similar way to $h_0(x)$, namely $h_1(x)/h_1(0)$ is a universal function of x/x_0. We are then led automatically to a GELL scaling theory for the second most singular amplitudes. This generalisation involves in addition to the n and g parameters of LL scaling, a third parameter that we denote by m. These three parameters g, n and m reflect the three scaling lengths pertinent to the problem; these are respectively x_0 for x, $h_0(0)$ for $h_0(x)$ and $h_1(0)$ for $h_1(x)$. At the end of this section we discuss how GELL scaling is modified in the event that the scaling length $h_1(0)=0$.

It is clear that scaling the higher-order correction functions $k_1(x)$, $h_2(x)$, ... in an analogous manner will lead to scaling theories for the third most singular amplitudes and higher. However, we have not thought it worthwhile to pursue these ideas beyond the second amplitudes.

We show in § 4 that ELL scaling is contained within GELL scaling as the special case $m = n$. Alternatively, ELL scaling can be viewed as arising when both $h_0(x)/h_0(0)$ *and* $h_1(x)/h_0(0)$ are universal functions of x/x_0; that is, the one scaling length, $h_0(0)$, is adequate for both scaling functions.

It is known that ELL scaling ($m = n$) holds exactly for the triangular, honeycomb and square lattices but not for the Kagomé lattice. In § 5 we show that for the Kagomé lattice the second most singular amplitudes are only given exactly by using GELL scaling with $m \neq n$. We calculate the exact values of g, n and m for the Kagomé lattice and show that the magnitudes of the exact values of all the second most singular amplitudes are all 1·398396% smaller than the ELL scaling predictions.

In § 6 we study the form of the scaling functions $h_0(x)$ and $h_1(x)$ for the Ising model in both the mean field and Bethe approximations, and also for the spherical model. We find that $h_0(x)/h_0(0)$ is a universal function of x/x_0 in all these cases. However, the mean field approximation for the Ising model is the only case for which $h_1(x)/h_1(0)$ is a universal function of x/x_0 and even then only in a rather trivial sense. Clearly ELL scaling and GELL scaling are not of general validity.

Finally, we summarise our results and conclusions in § 7.

2. Lattice–lattice scaling

The generalised law of corresponding states asserts that for various lattice models, including the Ising model, the most singular part of the free energy per site on lattice X, namely $f_X(t_X, h_X) = N_X^{-1} \ln Z_X$, is related to the most singular part of the free energy per site on Y, namely $f_Y(t_Y, h_Y)$, by the relation

$$n_X f_X(t_X, h_X) = n_Y f_Y(t_Y, h_Y) = f(t, h). \tag{2.1}$$

Here the reduced magnetic field variable $h = mH/kT$ is scaled according to

$$n_X h_X = n_Y h_Y = h \tag{2.2}$$

and the reduced temperature variable $t = (T - T_c)/T$ is scaled according to

$$g_X t_X = g_Y t_Y = t. \tag{2.3}$$

The law was first stated in precisely this form by Ritchie and Betts (1975). The earlier formulation of Betts *et al* (1971) used mH/kT_c as the reduced field variable and $(T - T_c)/T_c$ as the reduced temperature variable. For LL scaling the distinction between the two choices is immaterial but for ELL scaling to hold for the square, triangular and honeycomb Ising lattices it is important (Ritchie and Betts 1975) to use (as we do throughout this paper) mH/kT and $(T - T_c)/T$.

From the above law follow the LL scaling relations (Betts *et al* 1971) for the critical amplitudes of various thermodynamic quantities corresponding to the dominant singular term in their asymptotic behaviour close to the critical point. For example, if we write the spontaneous magnetisation (using the conventional notation) as

$$M_0(T) \sim B_0(-t)^\beta + B_1(-t)^{\beta+1} + \ldots \qquad (H = 0, t \to 0-), \tag{2.4}$$

then (2.1) yields the LL scaling relation

$$B_{0,X}/B_{0,Y} = (g_X/g_Y)^\beta. \tag{2.5}$$

Since B_0 and $\beta=\frac{1}{8}$ are known exactly for all two-dimensional lattices, the g-parameters may be computed exactly from this relation by taking the triangular lattice as a standard and setting $g_T=1$.

The n-parameters are simply related to the amplitude D_0 describing the shape of the critical isotherm

$$h_c \equiv mH/kT_c \sim D_0 M^\delta \qquad (t=0, H\to 0+) \tag{2.6}$$

by the LL scaling relation

$$D_{0,X}/D_{0,Y} = n_Y/n_X. \tag{2.7}$$

Unfortunately the amplitudes D_0 are not known exactly and hence (2.7) is unsuitable for the numerical determination of the n-parameters. However, the specific heat amplitudes defined by

$$C_{H=0}(T)/k \sim \phi^+(t) + A_0^+ t^{-\alpha} + A_1^+ t^{-\alpha+1} + \ldots \qquad (H=0, t\to 0+) \tag{2.8}$$

$$\sim \phi^-(t) + A_0^-(-t)^{-\alpha'} + A_1^-(-t)^{-\alpha'+1} + \ldots \qquad (H=0, t\to 0-) \tag{2.9}$$

where $\phi_1^\pm(t)$ are analytic functions of t, are known exactly in two dimensions. (Note that for the two-dimensional Ising model $\alpha=\alpha'=0$ and the terms $t^{-\alpha}, \ldots$ etc should be replaced by $-\ln t, \ldots$ etc.) One finds

$$A_{0,X}^+/A_{0,Y}^+ = (n_Y/n_X)(g_X/g_Y)^{2-\alpha} = A_{0,X}^-/A_{0,Y}^-, \qquad (\alpha=\alpha'), \tag{2.10}$$

from which exact values of the n-parameters can be obtained for all the two-dimensional lattices upon taking the triangular lattice as standard and setting $n_T=1$.

Now let us see what the LL scaling relations imply about the scaling function $h_0(x)$ in (1.1). Clearly, the spontaneous magnetisation corresponds to some point $x = -x_0(x_0>0)$ at which $h_0(-x_0)=0$ so that $H=0$ for $M\neq 0$. From (1.2)

$$M_0 = x_0^{-\beta}(-t)^\beta \tag{2.11}$$

so that by comparison with (2.4)

$$B_0 = x_0^{-\beta} \tag{2.12}$$

and hence

$$B_{0,X}/B_{0,Y} = (x_{0,Y}/x_{0,X})^\beta. \tag{2.13}$$

From (2.13) and (2.5) it follows that

$$g_X/g_Y = x_{0,Y}/x_{0,X}. \tag{2.14}$$

Now along the critical isotherm $x=0$, so that comparison of (1.1) and (2.6) gives

$$D_0 = h_0(0) \neq 0. \tag{2.15}$$

Hence, from (2.15) and (2.7) we get

$$n_X/n_Y = h_{0,Y}(0)/h_{0,X}(0). \tag{2.16}$$

Equations (2.14) and (2.16) express the fundamental n- and g-parameters of LL scaling theory in terms of basic parameters relating to the scaling function $h_0(x)$.

It is known (Griffiths 1967) that $h_0(x)$ has a convergent Taylor series expansion about $x=0$, namely

$$h_0(x) = \sum_{l=0}^{\infty} h_{0,l} x^l \tag{2.17}$$

where

$$h_{0,0}=h_0(0). \tag{2.18}$$

Substituting (1.2) into (2.17) and using (1.1), we obtain

$$h=\sum_{l=0}^{\infty} h_{0,l}M^{\delta-(l/\beta)}t^l. \tag{2.19}$$

On integrating with respect to M we obtain the singular behaviour of the higher-temperature derivatives of the free energy evaluated at the critical temperature,

$$\left.\frac{\partial^l f}{\partial t^l}\right|_{t=0} \sim -\frac{l!h_{0,l}}{[\delta+1-(l/\beta)]}M^{\delta+1-(l/\beta)}, \qquad (l=0,1,2,\dots). \tag{2.20}$$

Reverting (2.6) and using (2.15) gives

$$M\sim h_0(0)^{-1/\delta}h_c^{1/\delta} \qquad (t=0, H\to 0+). \tag{2.21}$$

Substituting (2.21) into (2.20) we find

$$\left.\frac{\partial^l f}{\partial t^l}\right|_{t=0} \sim A_l^0 h_c^{-\alpha_l} \qquad (t=0, H\to 0+) \tag{2.22}$$

where

$$A_l^0=(l!/\delta\alpha_l)h_{0,l}(h_0(0))^{\alpha_l}, \tag{2.23}$$

$$\alpha_l=\frac{l}{\Delta}-1-\frac{1}{\delta} \tag{2.24}$$

and we have written $\Delta=\beta\delta$. From (2.23) it follows on assuming that critical exponents are lattice independent for a given model and dimensionality that

$$\frac{A_{l,X}^0}{A_{l,Y}^0}=\frac{h_{0,l,X}}{h_{0,l,Y}}\left(\frac{h_{0,X}(0)}{h_{0,Y}(0)}\right)^{\alpha_l}=\frac{h_{0,l,X}}{h_{0,l,Y}}\left(\frac{n_Y}{n_X}\right)^{\alpha_l}. \tag{2.25}$$

Equating this to the LL scaling relation of Betts *et al* (1971), namely

$$\frac{A_{l,X}^0}{A_{l,Y}^0}=\left(\frac{g_X}{g_Y}\right)^l\left(\frac{n_Y}{n_X}\right)^{\alpha_l+1} \tag{2.26}$$

we get

$$\frac{h_{0,l,X}}{h_{0,l,Y}}=\left(\frac{n_Y}{n_X}\right)\left(\frac{g_X}{g_Y}\right)^l=\frac{h_{0,X}(0)}{h_{0,Y}(0)}\left(\frac{x_{0,Y}}{x_{0,X}}\right)^l \tag{2.27}$$

on using (2.14) and (2.16). Thus,

$$h_{0,l,X}x_{0,X}^l/h_{0,X}(0)=h_{0,l,Y}x_{0,Y}^l/h_{0,Y}(0)=c_{0,l} \tag{2.28}$$

where $c_{0,l}$ is a universal constant. Hence,

$$h_{0,l}=c_{0,l}h_0(0)/x_0^l \tag{2.29}$$

and substituting into (2.17) we finally get

$$H_0(x)\equiv h_0(x)/h_0(0)=\sum_{l=0}^{\infty} c_{0,l}(x/x_0)^l. \tag{2.30}$$

According to (2.30), $H_0(x)$ has a convergent Taylor expansion in powers of

$$u = x/x_0 \tag{2.31}$$

with coefficients $c_{0,l}$ which are lattice independent. Provided that $H_0(x)$ does not have a natural boundary of singularities in the complex x-plane, we may conclude that $H_0(x)$ *is a universal function over the entire physical range*, that is, real u such that $-1 \leq u < \infty$.

Alternatively, we may start from the large-x expansion of $h_0(x)$ (Griffiths 1967)

$$h_0(x) = \sum_{l=1}^{\infty} \eta_l x^{\beta(\delta+1-2l)} \tag{2.32}$$

which as shown by Domb and Hunter (1965) is equivalent to the $(2l-1)$th derivative of M with respect to H having a singularity of the form

$$\left(\frac{kT}{m}\right)^{2l-1} \frac{\partial^{2l-1} M}{\partial H^{2l-1}}\bigg|_{H=0} \sim C^{+}_{0,2l} t^{-\gamma-2(l-1)\Delta}, \qquad (H=0,\ t\to 0+), \tag{2.33}$$

where $l = 1, 2, 3, \ldots$. Now LL scaling theory gives

$$\frac{C^{+}_{0,2l,X}}{C^{+}_{0,2l,Y}} = \left(\frac{n_X}{n_Y}\right)^{2l-1} \left(\frac{g_Y}{g_X}\right)^{\gamma_{2l}}, \qquad (l = 1, 2, 3, \ldots) \tag{2.34}$$

where according to scaling theory

$$\gamma_{2l} = \gamma + 2(l-1)\Delta, \qquad (l = 1, 2, 3, \ldots). \tag{2.35}$$

By expressing the $C^{+}_{0,2l}$ in terms of the η_l and then using (2.34) it can be shown to any desired order that

$$\eta_l = b_{0,l} h_0(0)/x_0^{\beta(\delta+1-2l)}, \qquad (l = 1, 2, 3, \ldots) \tag{2.36}$$

where $b_{0,l}$ are universal constants. Substituting (2.36) into (2.32) gives the final result

$$H_0(x) = \sum_{l=1}^{\infty} b_{0,l} u^{\beta(\delta+1-2l)}, \tag{2.37}$$

so that once again we see that $H_0(x)$ is a universal function of $u = x/x_0$.

Below T_c, the analogues of (2.33) and (2.34) (Gaunt and Domb 1970, Betts *et al* 1971) yield a Taylor expansion for $H_0(x)$ about $u = -1$ of the form

$$H_0(x) = \sum_{l=1}^{\infty} d_{0,l} (u+1)^l, \tag{2.38}$$

with coefficients $d_{0,l}$ which are universal constants.

Finally we remark that there is nothing particularly significant about scaling the function $h_0(x)$ by the 'length' $h_0(0)$. The scaling length $h_0(a) \neq 0$ will do just as well, since if $h_0(x)/h_0(0)$ is a universal function of u then

$$\frac{h_0(x)}{h_0(a)} = \frac{h_0(x)/h_0(0)}{h_0(a)/h_0(0)} = \frac{H_0(u)}{H_0(a/x_0)} \tag{2.39}$$

is universal too. It follows that

$$n_X/n_Y = h_{0,Y}(0)/h_{0,X}(0) = h_{0,Y}(a)/h_{0,X}(a). \tag{2.40}$$

3. Generalised extended lattice–lattice scaling

To extend lattice–lattice scaling to the next most singular term we include the leading correction term in (1.3), namely

$$h = M^{\delta} h_0(x) + M^{\delta+(1/\beta)} h_1(x). \tag{3.1}$$

As we have seen in § 2, LL scaling for the dominant singular term implies that $h_0(x)/h_0(0)$ is a universal function of x/x_0. We now assume that

$$h_1(x)/h_1(0) \equiv H_1(u), \qquad (h_1(0) \neq 0) \tag{3.2}$$

is also a universal function of x/x_0. The scaling length $h_1(0)$ is no more special for $h_1(x)$ than was $h_0(0)$ for $h_0(x)$. An argument exactly analogous to that pertaining to (2.39) shows that any length such as $h_1(b) \neq 0$ is just as good. By analogy with (2.16), let us define a new scaling parameter by

$$m_X/m_Y = h_{1,Y}(0)/h_{1,X}(0) = h_{1,Y}(b)/h_{1,X}(b). \tag{3.3}$$

$H_1(u)$ will have universal expansions about $u = -1$, $u = 0$ and $u = \infty$ of the form (Domb 1971)

$$H_1(u) = \sum_{l=0}^{\infty} d_{1,l}(u+1)^l \tag{3.4}$$

$$= \sum_{l=0}^{\infty} c_{1,l} u^l \tag{3.5}$$

$$= \sum_{l=1}^{\infty} b_{1,l} u^{\beta(\delta+1-2l)+1} \tag{3.6}$$

which are the analogues of (2.38), (2.30) and (2.37), respectively.

First let us examine the leading correction term to the dominant behaviour given by (2.6) along the critical isotherm. Putting $x = 0$ in (3.1) we see that

$$h_c \sim D_0 M^{\delta} + D_1 M^{\delta+(1/\beta)}, \qquad (t = 0, H \to 0+), \tag{3.7}$$

where D_0 is given by (2.15) and

$$D_1 = h_1(0). \tag{3.8}$$

Hence

$$D_{1,X}/D_{1,Y} = h_{1,X}(0)/h_{1,Y}(0) = m_Y/m_X. \tag{3.9}$$

Now let us calculate the corrections to the spontaneous magnetisation. We require a solution of (3.1) for which $M_0 \neq 0$ when $H = 0$. As we saw in § 2, the zeroth-order term corresponds to $x = -x_0$. Now consider a small deviation from $-x_0$ to $-x_0 + \epsilon x_0$, where

$$M_0 = x_0^{-\beta}(-t)^{\beta}(1 + \beta\epsilon + \ldots). \tag{3.10}$$

To first order, we must solve the equation

$$0 = \epsilon x_0 h_0'(-x_0) + M_0^{1/\beta} h_1(-x_0) \tag{3.11}$$

giving (Domb and Gaunt 1971)

$$\epsilon = -x_0^{-2}(h_1(-x_0)/h_0'(-x_0))(-t). \tag{3.12}$$

By comparison with (2.4) we see that

$$B_1 = -\beta x_0^{-(2+\beta)}(h_1(-x_0)/h_0'(-x_0)). \tag{3.13}$$

From (2.38) and (3.4) it follows that

$$h_0'(-x_0) = d_{0,1}h_0(0)x_0^{-1} \tag{3.14}$$

and

$$h_1(-x_0) = d_{1,0}h_1(0) \tag{3.15}$$

so substituting in (3.13) gives

$$B_1 = -\frac{\beta}{x_0^{1+\beta}}\frac{d_{1,0}}{d_{0,1}}\frac{h_1(0)}{h_0(0)}. \tag{3.16}$$

Hence,

$$\frac{B_{1,X}}{B_{1,Y}} = \left(\frac{x_{0,Y}}{x_{0,X}}\right)^{1+\beta} \cdot \frac{h_{1,X}(0)}{h_{1,Y}(0)} \cdot \frac{h_{0,Y}(0)}{h_{0,X}(0)} \tag{3.17}$$

and on using (2.14), (2.16) and (3.3) this becomes

$$\frac{B_{1,X}}{B_{1,Y}} = \left(\frac{g_X}{g_Y}\right)^{1+\beta}\left(\frac{m_Y}{m_X}\right)\left(\frac{n_X}{n_Y}\right). \tag{3.18}$$

Now let us calculate the leading correction term for the high-temperature zero-field susceptibility $\chi_0(T)$. Substituting (2.37) and (3.6) into (3.1) and using (1.2) we obtain an expansion in odd powers of M, namely

$$mH/kT = (h_0(0)b_{0,1}x_0^{-\gamma}t^{\gamma} + h_1(0)b_{1,1}x_0^{-(\gamma+1)}t^{\gamma+1})M + \mathrm{O}(M^3) \tag{3.19}$$

to leading order. From (3.19) we find

$$(kT/m^2)\chi_0 \sim C_{0,2}^{+}t^{-\gamma} + C_{1,2}^{+}t^{-\gamma+1} \qquad (H=0, t\to 0+) \tag{3.20}$$

where

$$C_{0,2}^{+} = x_0^{\gamma}/(b_{0,1}h_0(0)) \tag{3.21}$$

and

$$C_{1,2}^{+} = -x_0^{\gamma-1}\frac{h_1(0)}{(h_0(0))^2}\frac{b_{1,1}}{b_{0,1}^2}. \tag{3.22}$$

From (3.21) we get

$$\frac{C_{0,2,X}^{+}}{C_{0,2,Y}^{+}} = \left(\frac{x_{0,X}}{x_{0,Y}}\right)^{\gamma}\frac{h_{0,Y}(0)}{h_{0,X}(0)} = \left(\frac{g_Y}{g_X}\right)^{\gamma}\left(\frac{n_X}{n_Y}\right) \tag{3.23}$$

which confirms (2.34) when $l=1$, while (3.22) gives

$$\frac{C_{1,2,X}^{+}}{C_{1,2,Y}^{+}} = \left(\frac{x_{0,X}}{x_{0,Y}}\right)^{\gamma-1}\left(\frac{h_{0,Y}(0)}{h_{0,X}(0)}\right)^2\frac{h_{1,X}(0)}{h_{1,Y}(0)} = \left(\frac{g_Y}{g_X}\right)^{\gamma-1}\left(\frac{n_X}{n_Y}\right)^2\left(\frac{m_Y}{m_X}\right) \tag{3.24}$$

for the leading correction term.

The low-temperature susceptibility may be dealt with by again considering a small deviation ϵx_0 from $-x_0$. Suppose one defines

$$(kT/m^2)\chi_0 \sim C_{0,2}^{-}(-t)^{-\gamma'} + C_{1,2}^{-}(-t)^{-\gamma'+1} \qquad (H=0, t\to 0-) \tag{3.25}$$

then (3.1), together with (2.38) and (3.4), and, of course, the scaling result $\gamma' = \gamma$ lead after some algebra to

$$C_{0,2}^{-} = \beta x_0^{\gamma}/(d_{0,1}h_0(0)) \tag{3.26}$$

$$C_{1,2}^{-} = \frac{\beta x_0^{\gamma-1}h_1(0)}{d_{0,1}^2(h_0(0))^2}\left(\frac{2d_{1,0}d_{0,2}}{d_{0,1}} - d_{1,1} - \frac{2-\gamma}{\beta}d_{1,0}\right). \tag{3.27}$$

Hence, we find

$$\frac{C_{0,2,X}^{-}}{C_{0,2,Y}^{-}} = \left(\frac{x_{0,X}}{x_{0,Y}}\right)^{\gamma}\frac{h_{0,Y}(0)}{h_{0,X}(0)} = \left(\frac{g_Y}{g_X}\right)^{\gamma}\frac{n_X}{n_Y} = \frac{C_{0,2,X}^{+}}{C_{0,2,Y}^{+}} \tag{3.28}$$

in agreement with Betts *et al* (1971), and in addition

$$\frac{C_{1,2,X}^{-}}{C_{1,2,Y}^{-}} = \left(\frac{x_{0,X}}{x_{0,Y}}\right)^{\gamma-1}\left(\frac{h_{0,Y}(0)}{h_{0,X}(0)}\right)^2\left(\frac{h_{1,X}(0)}{h_{1,Y}(0)}\right)$$

$$= \left(\frac{g_Y}{g_X}\right)^{\gamma-1}\left(\frac{n_X}{n_Y}\right)^2\left(\frac{m_Y}{m_X}\right) = \frac{C_{1,2,X}^{+}}{C_{1,2,Y}^{+}}. \tag{3.29}$$

The singular part of the zero-field free energy $f_0(T)$ can be studied by generalising Griffiths' arguments (Griffiths 1967) for the dominant singularity. The treatment is rather more detailed than that presented so far and we simply summarise the main results. The cases $\alpha > 0$ and $\alpha = 0$ must be considered separately. When $\alpha = \alpha' = 0$ we find

$$-f_0(T) = F_0^{\pm}t^2 \ln|t| + F_1^{\pm}|t|t^2 \ln|t|, \qquad (H = 0, t \to 0\pm) \tag{3.30}$$

where

$$F_0^{+} = c_{0,2}\beta h_0(0)x_0^{-2} = F_0^{-} \tag{3.31}$$

and

$$F_1^{+} = c_{1,3}\beta h_1(0)x_0^{-3} = F_1^{-}. \tag{3.32}$$

Hence,

$$\frac{F_{0,X}^{+}}{F_{0,Y}^{+}} = \left(\frac{x_{0,Y}}{x_{0,X}}\right)^2\frac{h_{0,X}(0)}{h_{0,Y}(0)} = \left(\frac{g_X}{g_Y}\right)^2\left(\frac{n_Y}{n_X}\right) = \frac{F_{0,X}^{-}}{F_{0,Y}^{-}} \tag{3.33}$$

and

$$\frac{F_{1,X}^{+}}{F_{1,Y}^{+}} = \left(\frac{x_{0,Y}}{x_{0,X}}\right)^3\frac{h_{1,X}(0)}{h_{1,Y}(0)} = \left(\frac{g_X}{g_Y}\right)^3\left(\frac{m_Y}{m_X}\right) = \frac{F_{1,X}^{-}}{F_{1,Y}^{-}}. \tag{3.34}$$

When $\alpha' = \alpha > 0$, these relations generalise to

$$\frac{F_{0,X}^{+}}{F_{0,Y}^{+}} = \left(\frac{g_X}{g_Y}\right)^{2-\alpha}\left(\frac{n_Y}{n_X}\right) = \frac{F_{0,X}^{-}}{F_{0,Y}^{-}} \tag{3.35}$$

and

$$\frac{F_{1,X}^{+}}{F_{1,Y}^{+}} = \left(\frac{g_X}{g_Y}\right)^{3-\alpha}\left(\frac{m_Y}{m_X}\right) = \frac{F_{1,X}^{-}}{F_{1,Y}^{-}}, \tag{3.36}$$

respectively. Since the specific heat is obtained essentially by differentiating $f_0(T)$ twice with respect to T, it is easily seen that (3.35) is equivalent to (2.10). However, the amplitude of the next most singular term in the specific heat does not scale because of an additive contribution from the leading-order term which arises during differentiation.

To summarise, by assuming $H_1(x)$ is a universal function of x/x_0, we have derived lattice–lattice scaling formulae for the next most singular term of the zero-field free energy and susceptibility at both low and high temperatures, of the spontaneous magnetisation and of the shape of the critical isotherm. Analogous relations may also be derived for the higher-field derivatives of the free energy at both low and high temperatures evaluated in zero field and for the higher-temperature derivatives evaluated at the critical temperature. For example, on allowing for the next term, (2.22) becomes

$$\left.\frac{\partial^l f}{\partial t^l}\right|_{t=0} \sim A_l^0 h_c^{-\alpha_l} + A_l^1 h_c^{-\beta_l}, \qquad (t=0, H\to 0+), \tag{3.37}$$

with

$$\beta_l = \frac{l-1}{\Delta} - 1 - \frac{1}{\delta}. \tag{3.38}$$

We find that A_l^1 scales in the following way

$$\frac{A_{l,X}^1}{A_{l,Y}^1} = \left(\frac{g_X}{g_Y}\right)^l \left(\frac{m_Y}{m_X}\right) \left(\frac{n_Y}{n_X}\right)^{\beta_l}. \tag{3.39}$$

We conclude this section by pointing out how GELL scaling is modified if $h_1(0)=0$. Such a discussion is pertinent since $h_1(0)=0$ is a definite possibility for both two- and three-dimensional lattices (Domb and Gaunt 1971). In this case, we must scale $h_1(x)$ by some non-zero length $h_1(b)$. GELL scaling relations for both high- and low-temperature amplitudes will clearly be unaffected since they are computed from the expansions of $h_1(x)$ about $x=\infty$ and $x=-x_0$, respectively. However, we expect some modification for amplitudes defined at T_c since putting $x=0$ in (1.3) now gives, in contrast to (3.7),

$$h_c \sim h_0(0) M^\delta + k_1(0) M^{\delta+(\gamma/\beta)}, \qquad (t=0, H\to 0+) \tag{3.40}$$

provided $k_1(0)\neq 0$. For greater generality, let us assume that not only $h_1(0)=0$ but that the first L derivatives of $h_1(x)$ at $x=0$ are zero also. We then find that (3.37) holds only for $l\geqslant L+1$; for $l\leqslant L$ we get

$$\left.\frac{\partial^l f}{\partial t^l}\right|_{t=0} \sim A_l^0 h_c^{-\alpha_l} + A_l^1 h_c^{-\gamma_l} \qquad (t=0, H\to 0+) \tag{3.41}$$

with

$$\gamma_l = \frac{l-\gamma}{\Delta} - 1 - \frac{1}{\delta} \qquad (l\leqslant L) \tag{3.42}$$

and

$$\frac{A_{l,X}^1}{A_{l,Y}^1} = \left(\frac{g_X}{g_Y}\right)^l \left(\frac{k_Y}{k_X}\right) \left(\frac{n_Y}{n_X}\right)^{\gamma_l}, \qquad (l\leqslant L). \tag{3.43}$$

These last three results were obtained by assuming (in the same spirit as before) that $k_1(0)$ is a suitable scaling length for $k_1(x)$. Hence, $k_1(x)/k_1(0)$ is a universal function of x/x_0 and we are led to introduce a fourth scaling parameter, k, defined in analogy with m and n by

$$k_X/k_Y = k_{1,Y}(0)/k_{1,X}(0). \tag{3.44}$$

Consequently, in the special case that

$$h_1(x)\Big|_{x=0} = \frac{dh_1(x)}{dx}\Big|_{x=0} = \ldots = \frac{d^L h_1(x)}{dx^L}\Big|_{x=0} = 0, \tag{3.45}$$

we have a lattice–lattice scaling theory involving four parameters, g, n, m and k, although no more than three of them can occur in any one expression.

4. Extended lattice–lattice scaling

Let us set

$$m_X = n_X, \qquad m_Y = n_Y \tag{4.1}$$

in the above expressions. Thus, we obtain

$$\frac{F^+_{1,X}}{F^+_{1,Y}} = \left(\frac{g_X}{g_Y}\right)^{3-\alpha}\left(\frac{n_Y}{n_X}\right) = \frac{F^-_{1,X}}{F^-_{1,Y}}, \tag{4.2}$$

$$\frac{C^+_{1,2,X}}{C^+_{1,2,Y}} = \left(\frac{g_Y}{g_X}\right)^{\gamma-1}\left(\frac{n_X}{n_Y}\right) = \frac{C^-_{1,2,X}}{C^-_{1,2,Y}}, \tag{4.3}$$

$$\frac{B_{1,X}}{B_{1,Y}} = \left(\frac{g_X}{g_Y}\right)^{1+\beta} \tag{4.4}$$

for the zero-field free energy, zero-field susceptibility and spontaneous magnetisation, respectively, and

$$\frac{A^1_{l,X}}{A^1_{l,Y}} = \left(\frac{g_X}{g_Y}\right)^{l}\left(\frac{n_Y}{n_X}\right)^{1+\beta_l}, \tag{4.5}$$

$$D_{1,X}/D_{1,Y} = n_Y/n_X \tag{4.6}$$

along the critical isotherm. These relations are identical with the ELL scaling results (Guttmann 1974, Ritchie and Betts 1975), which are known to hold exactly for the triangular, honeycomb and square lattices. It seems that these lattices correspond to the special case of $m = n$. This implies through (2.16) and (3.3) that $h_1(0)/h_0(0)$ is a universal constant, ψ say, so that

$$h_1(x)/h_0(0) = H_1(x)(h_1(0)/h_0(0))$$

is a universal function of x/x_0. In other words, ELL scaling implies that only two scaling lengths, rather than the usual three, are required, namely, x_0 for x, and $h_0(0)$, for example, for both $h_0(x)$ *and* $h_1(x)$. We can now use (3.7), (3.8) and (2.15) to make the simple prediction that along the critical isotherm

$$h_c \sim D_0 M^{\delta}(1 + \psi M^{1/\beta}), \qquad (t = 0, H \to 0+), \tag{4.7}$$

that is, the leading correction term is the same for all three lattices.

5. Kagomé lattice

It was shown by Ritchie and Betts (1975) that although ELL scaling holds for the triangular, honeycomb and square lattices, it does not hold for the Kagomé lattice. In table 1 we give the exact values of the second most singular amplitudes for the zero-field free energy ($F_1^{\pm}$), spontaneous magnetisation (B_1) and zero-field susceptibility ($C_{1,2}^{\pm}$). We also give for comparison the ELL scaling predictions. The interesting point to note is that their magnitudes are all 1·3984% larger than the magnitudes of the exact values. Indeed it was the study of the implications of this constant discrepancy that provided the initial motivation for this work.

Table 1. Exact values and ELL scaling predictions for the second most singular amplitudes for the Kagomé lattice.

Amplitude	Exact value	ELL scaling
$F_1^+ = F_1^-$	0·163 965 041	0·166 257 920
B_1	−0·475 945 821	−0·482 601 422
$C_{1,2}^+$	0·086 936 625	0·088 152 343
$C_{1,2}^-$	−0·002 306 400	−0·002 338 652

We have seen that ELL scaling implies $m = n$. If we allow m and n to differ, then the amplitudes of the second most singular term for the Kagomé lattice are readily accounted for by the GELL scaling theory that we developed in § 3. The g- and n-parameters for the Kagomé lattice may be calculated as before by using the exactly available amplitudes of the most singular term giving (Guttmann 1974)

$$g_K = 1{\cdot}260\,958\,918, \qquad n_K = 1{\cdot}652\,973\,376. \tag{5.1}$$

To calculate m_K we may use, for example, the GELL scaling result (3.18) which for the triangular–Kagomé pair becomes

$$B_{1,T}/B_{1,K} = m_K/(n_K g_K^{9/8}) \tag{5.2}$$

where we have set $m_T = n_T = g_T = 1$ (taking the triangular lattice as the standard) and $\beta = \frac{1}{8}$. Using the exact values of $B_{1,K}$ and $B_{1,T}$ ($=-0{\cdot}371\,791\,998\,8\ldots$) quoted in table 1 and by Guttmann (1975), respectively, we find

$$m_K = 1{\cdot}676\,088\,455. \tag{5.3}$$

The exact values of all the other amplitudes in table 1, namely $C_{1,2}^{\pm}$ and $F_1^{\pm}$, could now be predicted using g_K, n_K and m_K given by (5.1) and (5.3) and the appropriate GELL scaling results (3.29) and (3.34).

Thus we have found

$$m_K/n_K = 1{\cdot}013\,983\,955 \tag{5.4}$$

in contrast to

$$m_X/n_X = 1 \tag{5.5}$$

for the triangular, honeycomb and square lattices. Hence, for the Kagomé lattice, m/n exceeds its value for the other three lattices by $1{\cdot}398\,395\,5\ldots\,\%$. This is the source of the constant discrepancy noted in table 1 for the amplitudes. To see this, consider the ratio

$$R_{\rm K}=\frac{|A_{\rm K}|^{\rm ELL}-|A_{\rm K}|^{\rm exact}}{|A_{\rm K}|^{\rm exact}}=\frac{|A_{\rm K}|^{\rm ELL}}{|A_{\rm K}|^{\rm GELL}}-1, \tag{5.6}$$

where A is any of the second most singular amplitudes considered in § 3. For example, consider the susceptibility amplitude $C^{+}_{1,2}$. From (3.24)

$$C^{+({\rm GELL})}_{1,2,{\rm K}}/C^{+}_{1,2,{\rm T}}=n^{2}_{\rm K}/(m_{\rm K}g^{3/4}_{\rm K}) \tag{5.7}$$

and from (4.3)

$$C^{+({\rm ELL})}_{1,2,{\rm K}}/C^{+}_{1,2,{\rm T}}=n_{\rm K}/g^{3/4}_{\rm K}, \tag{5.8}$$

so that

$$R_{\rm K}=(m_{\rm K}/n_{\rm K})-1=1{\cdot}398\,395\,5\ldots\%. \tag{5.9}$$

It is easily verified using our earlier results that the same value of $R_{\rm K}$ is obtained for all the second most singular amplitudes.

As pointed out by Guttmann (1977), it is noteworthy that

$$-C^{+}_{1,2}/C^{-}_{1,2}=37{\cdot}693\,652=C^{+}_{0,2}/C^{-}_{0,2} \tag{5.10}$$

for the Kagomé lattice as well as for the triangular, honeycomb and square lattices. This follows immediately from our results; from (3.21) and (3.26) we find

$$C^{+}_{0,2}/C^{-}_{0,2}=d_{0,1}/\beta b_{0,1} \tag{5.11}$$

and from (3.22) and (3.27)

$$-C^{+}_{1,2}/C^{-}_{1,2}=b_{1,1}d^{3}_{0,1}b^{-2}_{0,1}[2\beta d_{0,2}d_{1,0}-\beta d_{0,1}d_{1,1}-(2-\gamma)d_{0,1}d_{1,0}]. \tag{5.12}$$

The right-hand sides of (5.11) and (5.12) are both universal functions, so if the left-hand sides are equal for one lattice they will be equal for all four lattices including the Kagomé lattice.

6. Scaling functions for exactly solvable problems

We naturally wondered if the scaling functions for any of the exactly solvable problems have the form proposed in this work. In this section, therefore, we study the mean-field and Bethe approximations for the Ising model, and the spherical model.

We shall look first at the mean-field approximation for the Ising model to emphasise that its equation of state can be expressed in the form (1.3). As shown by Domb (1971), this form is most readily derived by starting from the equation

$$M=\tanh\left(\frac{T_{\rm c}M}{T}+\frac{mH}{kT}\right)$$

giving

$$h_{\rm c}=M^{3}h_{0}(x)+M^{5}h_{1}(x)+M^{7}h_{2}(x)+\ldots \tag{6.1}$$

where

$$x=[(T-T_{\rm c})/T_{\rm c}]M^{-2} \tag{6.2}$$

and

$$h_0(x)=x+\tfrac{1}{3}, \qquad h_1(x)=\tfrac{1}{3}x+\tfrac{1}{5}, \qquad h_2(x)=\tfrac{1}{5}x+\tfrac{1}{7}, \qquad \ldots \tag{6.3}$$

so that all the $h_i(x)$ are linear. It should be noted however that the definition of x used by Domb, namely (6.2), is not the same as our definition (1.2). In addition, the expansion (6.1) is of mH/kT_c but in (1.3) of mH/kT, as discussed earlier. Transforming to the new variables we find

$$h_0(x)=x+\tfrac{1}{3}, \qquad h_1(x)=\tfrac{1}{5}, \qquad h_2(x)=\tfrac{1}{7}, \qquad \ldots \tag{6.4}$$

so that $h_0(x)$ is linear and all the other $h_i(x)$ are constants. Clearly $x_0=\frac{1}{3}$ and so

$$H_0(u)=1+u \tag{6.5}$$

$$H_i(u)=1 \qquad (i=1,2,3,\ldots). \tag{6.6}$$

The mean-field approximation ignores the lattice structure completely by focusing attention on a single spin. The fact that $h_i(x)/h_i(0)$ for all $i=0,1,2,\ldots$ is the same function of x/x_0 for all lattices is not therefore particularly illuminating.

In the Bethe approximation for the Ising model, the lattice structure enters in a simple way through the lattice coordination number $q=\sigma+1$. The magnetisation M is given by Domb (1960) as

$$M=(1-\mu_1^2)/(1+2\mu_1 z+\mu_1^2) \tag{6.7}$$

where μ_1 is an intermediate variable defined implicitly through the relation

$$\mu_1/\mu=(\mu_1+z)^\sigma/(1+z\mu_1)^\sigma. \tag{6.8}$$

Here μ and z are the usual Ising variables

$$\mu=\exp(-2mH/kT), \qquad z=\exp(-2J/kT). \tag{6.9}$$

The critical temperature is given in the Bethe approximation by

$$z_c=\exp(-2J/kT_c)=1-2q^{-1}. \tag{6.10}$$

After some fairly lengthy algebraic manipulation, (6.7) and (6.8) can be recast in the form

$$h=M^3h_0(x)+M^5h_1(x)+M^7h_2(x)+\ldots \tag{6.11}$$

with

$$h_0(x)=\tfrac{1}{2}qz_c[\alpha x+\tfrac{1}{6}(1-z_c^2)] \tag{6.12}$$

$$h_1(x)=\tfrac{1}{2}qz_c[\tfrac{1}{2}\alpha^2x^2+\tfrac{1}{2}\alpha(1-z_c^2)x+\tfrac{1}{40}(7-3z_c^2)(1-z_c^2)] \tag{6.13}$$

$$\vdots$$

where $\alpha=2J/kT_c$. From (6.12) we see that

$$x_0=(1-z_c^2)/6\alpha \tag{6.14}$$

$$h_0(0)=\tfrac{1}{12}qz_c(1-z_c^2) \tag{6.15}$$

and from (6.13)

$$h_1(0)=\tfrac{1}{80}qz_c(7-3z_c^2)(1-z_c^2). \tag{6.16}$$

Hence,

$$H_0(u)=(1+u) \tag{6.17}$$

and

$$H_1(u)=1+\frac{10}{3}\frac{q-1}{q^2+3q-3}u+\frac{5}{9}\frac{q-1}{q^2+3q-3}u^2. \tag{6.18}$$

We see that $H_0(u)$ is the same universal function of u as in the mean-field approximation, (6.5). However, $H_1(u)$ is clearly not a universal function of u. We have checked that alternative definitions of reduced field and temperature do not give rise to a simpler form. In the limit of $q\to\infty$, the Bethe approximation should reduce to the mean-field approximation (Fisher and Gaunt 1964) and indeed from (6.18) we see that

$$\lim_{q\to\infty} H_1(u)=1$$

in agreement with (6.6).

For the spherical model, the equation of state and its correction terms have been investigated for the body-centred cubic lattice only (Joyce 1972, see also Domb 1971). Transforming to our reduced field and temperature variables we find

$$h=M^5h_0(x)+M^7h_1(x)+M^9h_2(x)+\ldots \tag{6.19}$$

where

$$h_0(x)=b_1K_c^3(1+x)^2 \tag{6.20}$$

$$h_1(x)=b_2K_c^4(1+x)^3-b_1K_c^3x(1+x)(3+x). \tag{6.21}$$

Here b_1, b_2 and K_c are constants whose values depend upon the particular lattice under consideration. It follows from (6.20) that $x_0=1$ and hence

$$H_0(u)=(1+u)^2 \tag{6.22}$$

$$H_1(u)=(1+u)^3-(b_1/b_2K_c)u(1+u)(3+u). \tag{6.23}$$

Notice that all the three cases we have considered have the same functional form for H_0, namely

$$H_0(u)=(1+u)^\gamma. \tag{6.24}$$

Although we cannot be sure, since we only have the results for one lattice, it appears that $H_0(u)$ for the spherical model will be a universal function of u but that $H_1(u)$ will not be. Thus the spherical model and the mean-field approximation for the Ising model are alike in this respect. This breakdown of GELL scaling also occurs for the three-dimensional Ising model as reported in § 1.

Finally, note that $h_1(x)$ in (6.21) can be written

$$h_1(x)=h_1(0)U(u)+h_0(0)V(u), \tag{6.25}$$

where

$$U(u)=(1+u)^3, \qquad V(u)=-u(1+u)(3+u) \tag{6.26}$$

are possibly universal functions. This observation together with (6.30) prompted us to study the implications of the more general form

$$h_1(x) = h_1(0)U(u) + h_0(0)\left(\frac{h_1(0)}{h_0(0)}\right)^k V(u) \tag{6.27}$$

where $U(u)$ and $V(u)$ are universal functions of u, and k is an arbitrary constant. When $U(u) \equiv 0$, (6.27) implies ELL scaling since then $h_1(0)/h_0(0)$ is universal. If, on the other hand, $V(u) \equiv 0$ or $k = 1$ then $h_1(x)/h_1(0)$ is universal and we get GELL scaling. Suppose however that $U(u) \not\equiv 0$ and $V(u) \not\equiv 0$. There are then two cases to consider. Firstly, if $V(0) \neq 0$ substitution of $x = 0$ into (6.27) shows that $h_1(0)/h_0(0)$ is universal and this case therefore gives ELL scaling. Second, if $V(0) = 0$ it is easily shown (by the methods outlined earlier) that in general (6.27) does not lead to scaling behaviour. This case corresponds to the spherical model for which $k = 0$ and $V(u)$ is given by (6.26). However, we can still get ELL scaling if either $h_1(0)/h_0(0)$ is universal or if $U(u)$ happens to have the unlikely expansion

$$U(u) = -\sum_{l=0}^{\infty} \beta_l c_{0,l} u^l \tag{6.28}$$

where β_l and $c_{0,l}$ are defined by (3.38) and (2.30) respectively. In the latter case a series of cancellations occurs and one can show that the amplitudes A_l^1, for example, defined by (3.37) scale like

$$\frac{A_{l,X}^1}{A_{l,Y}^1} = \left(\frac{g_X}{g_Y}\right)^l \left(\frac{m_Y}{m_X}\right)^k \left(\frac{n_Y}{n_X}\right)^{\beta_l + 1 - k}. \tag{6.29}$$

It then follows that the ratio R_X, defined in analogy with (5.6) is given by

$$R_X = \left(\frac{m_X}{n_X}\right)^k - 1, \tag{6.30}$$

which reduces correctly when $k = 1$ to give (5.9). Indeed it was while studying the implications of an R of this form that we were first led to consider an $h_1(x)$ of the form given in (6.27).

7. Summary

In this paper we have explored the connection between LL scaling and the scaled equation of state of the Ising model and related models of phase transitions. All extensions and generalisations are subsequently formulated under the assumption that the correction to scaling exponent is unity above and below T_c. It has been shown that LL scaling requires $h_0(x)/h_0(0)$ to be a universal function of x/x_0. If, in addition, $h_1(x)/h_0(0)$ is a universal function of x/x_0 then ELL scaling results.

All the usually studied lattice models of phase transitions with short-range forces appear to satisfy LL scaling. However, only the Ising model on the square, triangular and honeycomb lattices, and (trivially) the Ising model in the mean-field approximation are known to satisfy ELL scaling. (Our knowledge of most other models is too incomplete to allow ELL scaling to be tested.) In examining the breakdown of ELL scaling for the Ising model on the Kagomé lattice, we are led to introduce GELL scaling, which requires $h_1(x)/h_1(0)$ to be a universal function of x/x_0. It is then apparent that ELL scaling is a special case of GELL scaling, corresponding to

$h_1(0)/h_0(0)$ being a universal constant. If $h_1(0)$ vanishes, then $h_1(x)$ must be scaled by $h_1(b) \neq 0$. The theory for this situation is also established.

An illuminating way to distinguish between these various theories is in terms of the number of distinct scaling lengths that enter the problem, apart from x_0 which always scales x. For LL scaling and ELL scaling there is but one scaling length $h_0(0)$, which scales both $h_0(x)$ and $h_1(x)$. For GELL scaling there are two scaling lengths, $h_0(0)$ to scale $h_0(x)$ and $h_1(0)$ to scale $h_1(x)$, unless $h_1(0)=0$ when there are three scaling lengths, but no more than two are needed for the discussion of any particular amplitude ratio.

The Ising model on the Kagomé lattice is the only non-trivial example known to satisfy GELL scaling. We have shown that in the Bethe approximation the Ising model does not satisfy GELL scaling, and that the spherical model is most unlikely to satisfy GELL scaling. We have also reported on our (unpublished) series analyses which strongly suggest that the three-dimensional Ising model does not satisfy GELL scaling either. This extends earlier work showing that the three-dimensional Ising model does not satisfy ELL scaling.

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